Singularly Continuous

A.3 Random Variable with Neither Density nor Probability Mass Function

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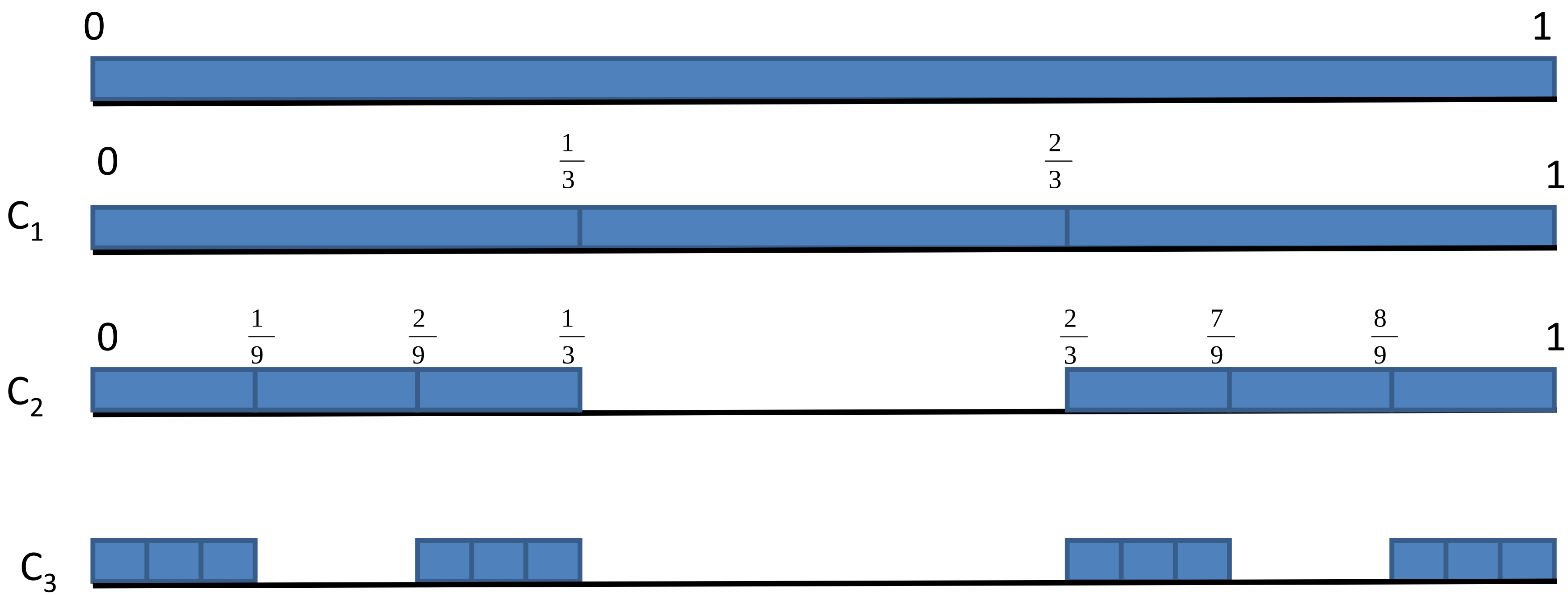
Singularly Continuous

- A singularly continuous function is a continuous function that has a zero derivative almost everywhere.
- The distributions which are continuous but with no density function are said to be singular.
- An example is the Cantor function.

Constructing the Cantor Set



Continue this process…

$$C = \bigcap_{k=1}^{\infty} C_k$$

The length of the Cantor Set

- The first set removed is 1/3.
- The second set removed is 2/9.
- The third set removed is 4*1/27=4/27.
- The kth set removed is $\frac{1}{3} * (\frac{2}{3})^{k-1}$
- The total length removed is $\frac{1}{3}\sum_{1}^{\infty}(\frac{2}{3})^{k-1} = 1$
- So the Cantor set, the set of points not removed, has zero "length".
- Lebesgue measure of the Cantor set is zero.

The Probability of Cantor Set

- C_1 is $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ which has two pieces, each with probability 1/3, and the whole set C_1 has probability 2/3.
- C_2 is $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ which has four pieces, each with probability 1/9, and the whole set C_1 has probability 4/9.
- At stage k, we have a set C_k that has 2^k pieces, each with probability $1/3^k$, and the whole set C_k has probability $(2/3)^k$.

$$\Rightarrow \quad \mathrm{P}(C) = \lim_{k \to \infty} \mathrm{P}(C_k) = \lim_{k \to \infty} \left(\frac{2}{3}\right)^k = 0$$

Random Variable

- Fro n=1,2,..., we define $Y_n(\omega) = \begin{cases} 1, \text{if } \omega_n = H \\ 0, \text{if } \omega_n = T \end{cases}$
- The Probability for head on each toss is p=1/2.
- Define the random variable $Y = \sum_{n=1}^{\infty} \frac{2Y_n}{3^n}$

$$Y = \sum_{n=1}^{\infty} \frac{2Y_n}{3^n} = \frac{2Y_1}{3} + \frac{2Y_2}{3^2} + \frac{2Y_3}{3^3} + \ldots$$

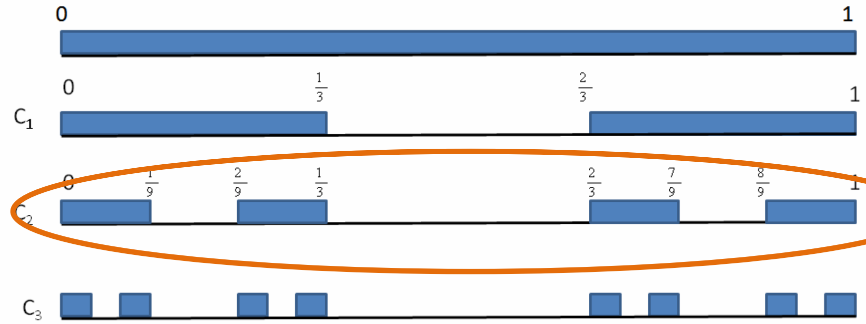
- Case 1.1: $Y_1=0$ with probability = 1/2
  - Minimum: $Y_2=Y_3=\ldots=0 \Rightarrow Y=0$
  - Maximum: $Y_2=Y_3=\ldots=1 \Rightarrow Y=1/3$

$\Rightarrow \quad 0 \le Y \le \frac{1}{3}$

- Case 1.2: $Y_1=1$ with probability = 1/2
  - Minimum: $Y_2=Y_3=\ldots=0 \Rightarrow Y=2/3$
  - Maximum: $Y_2=Y_3=\ldots=1 \Rightarrow Y=1$

$\Rightarrow \quad \frac{2}{3} \le Y \le 1$

$$Y=\sum_{n=1}^{\infty}\frac{2Y_n}{3^n}=\frac{2Y_1}{3}+\frac{2Y_2}{3^2}+\frac{2Y_3}{3^3}+\ldots$$

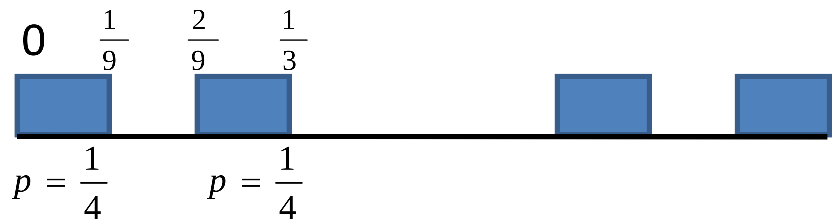


- Case 2.1: Y_1=0 and Y_2=0 with probability = ¼

  $\Rightarrow \quad 0 \le Y \le \frac{1}{9}$

- Case 2.2: Y_1=0 and Y_2=1 with probability = ¼

  $\Rightarrow \quad \frac{2}{9} \le Y \le \frac{1}{3}$



- …….
- When we consider the first n tosses we see that the random variable Y takes values in the set C_n.
- $Y=\sum_{n=1}^{\infty}\frac{2Y_n}{3^n}$ can only take values in the Cantor set $C=\bigcap_{n=1}^{\infty}C_n$.

Does Y have a density?

- If Y had a density ***f***…
- The complement of the Cantor set:

  f=0
- The Cantor set C:

  C has zero Lebesgue measure, i.e. P(C)=0
- So ***f*** is almost everywhere zero and $\int_0^1 f(x)dx = 0$
- The function f would not integrate to one, as is required of a density.

Does Y have a probability mass function?

- If Y had a probability mass function…

$\Rightarrow$For some number $x \in C$ we would have

$P(Y = x) > 0$

- If *x* is not of the form $\frac{k}{3^n}$, for some positive integers k and n
  - *x* has a unique base-three expansion $x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}, x_n = 0,1,2.$
- If x is of the form $\frac{k}{3^n}$, for some positive integers k and n
  - x has two base-three expansions.

$$\frac{7}{9} = \frac{2}{3} + \frac{1}{9} + \frac{0}{27} + \frac{0}{81} + \frac{0}{243} + \cdots = \frac{2}{3} + \frac{0}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$$

Does Y have a probability mass function? (cont.)

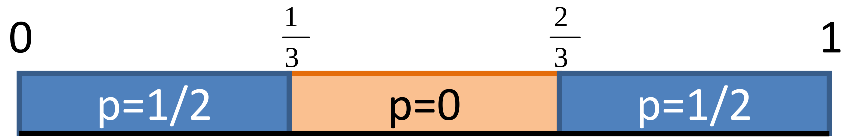
- In either case, there are at most two choices of $\omega \in \Omega_\infty$ for which $Y(\omega) = x$
- In other words, the set $\{\omega \in \Omega;\ Y(\omega) = x\}$ has either one or two elements.
- The probability of a set with one element is zero and the probability of a set with two elements is 0+0=0.

$\Rightarrow$ $P\{Y = x\} = 0$

$\Rightarrow$ Y cannot have a probability mass function.

Cumulative Distribution Function

- The cumulative distribution function $\mathrm{F(x) = P\{Y \leq x\}}$ satisfies

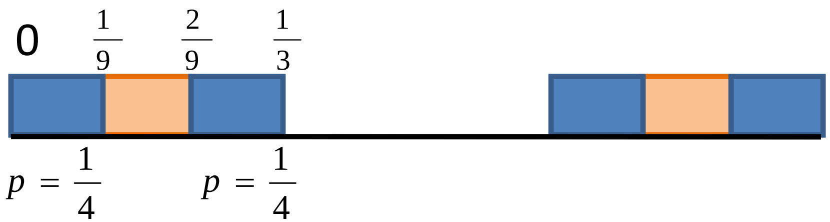


$$\mathrm{F(0) = 0, F(1) = 1, F(x) = \frac{1}{2} \text{ for } \frac{1}{3} \leq x \leq \frac{2}{3},}$$

$$\mathrm{F(x) = \frac{1}{4} \text{ for } \frac{1}{9} \leq x \leq \frac{2}{9}, F(x) = \frac{3}{4} \text{ for } \frac{7}{9} \leq x \leq \frac{8}{9},}$$

$$\mathrm{F(x) = \frac{1}{8} \text{ for } \frac{1}{27} \leq x \leq \frac{2}{27}, F(x) = \frac{3}{8} \text{ for } \frac{7}{27} \leq x \leq \frac{8}{27},}$$

$$\mathrm{F(x) = \frac{5}{8} \text{ for } \frac{19}{27} \leq x \leq \frac{20}{27}, F(x) = \frac{7}{8} \text{ for } \frac{25}{27} \leq x \leq \frac{26}{27},}$$



Cumulative Distribution Function

- $P\{Y = x\} = 0$ for every x, *F* is continuous.
- $F'(x) = 0$ for every $x \in [0,1] \setminus C$, which is almost every $x \in [0,1]$.
- A non-constant continuous function whose derivative is almost everywhere zero is said to be *singularly continuous*.

A Singularly Continuous Function

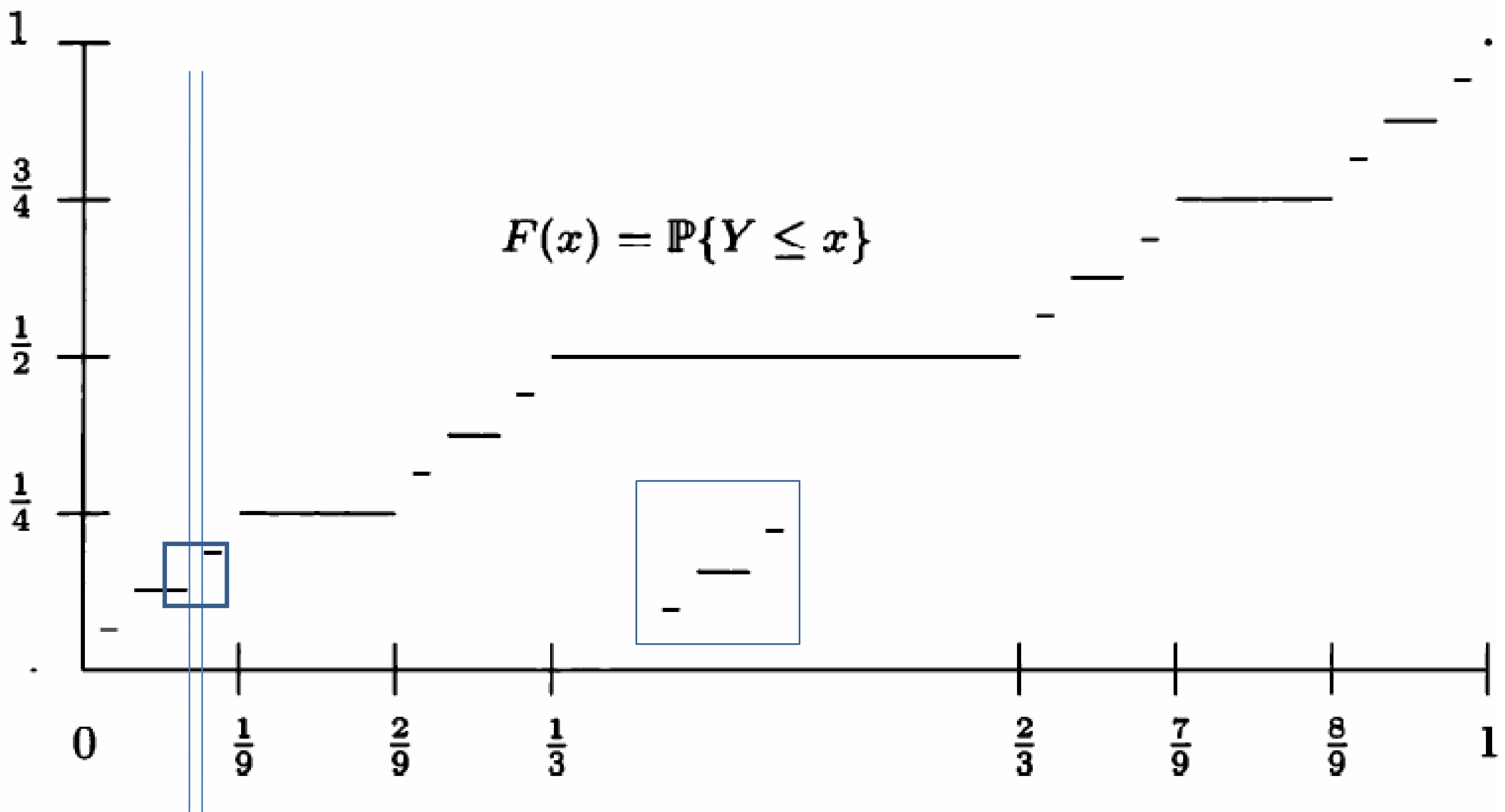
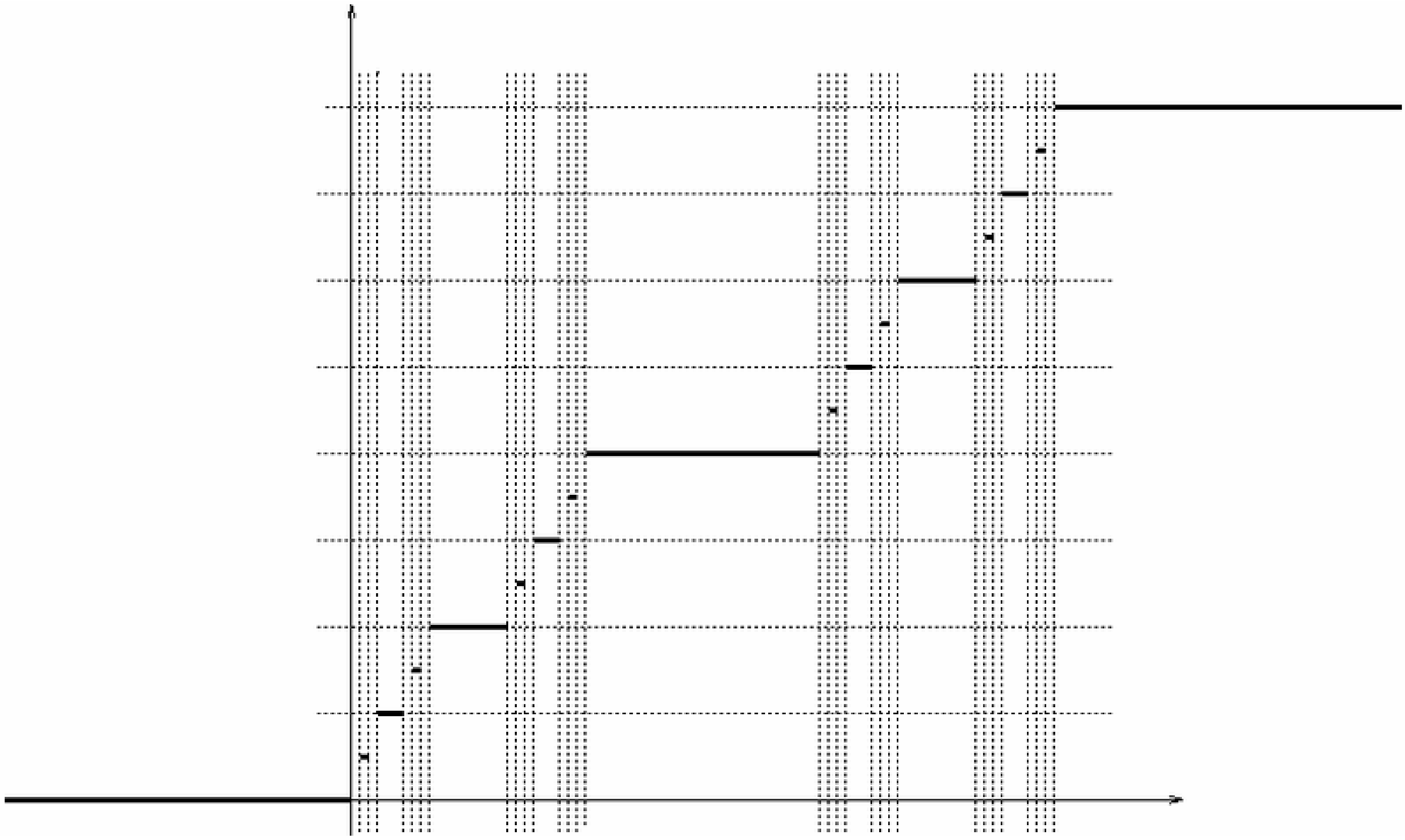


Fig. A.3.1. A singularly continuous function.



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